Physics 218B **Winter 2012**

Problem Set V: Due Thursday, March 15.

1.) Consider a fluid in hydrostatic equilibrium with a vertical entropy gradient $\partial S/\partial z < 0$. Take $\underline{g} = -g\hat{z}$.

a.) Starting from the basic equations, derive the growth rate of ideal Rayleigh-Bernard instability. You will find it helpful to relate the density perturbation $\tilde{\rho}/\rho_0$ to the temperature perturbation $\tilde{T}/T_0$ by exploiting the fact that the instability develops slowly in comparison to the sound transit time across a cell. Relate your result to the Schwarzschild criterion discussed in class.

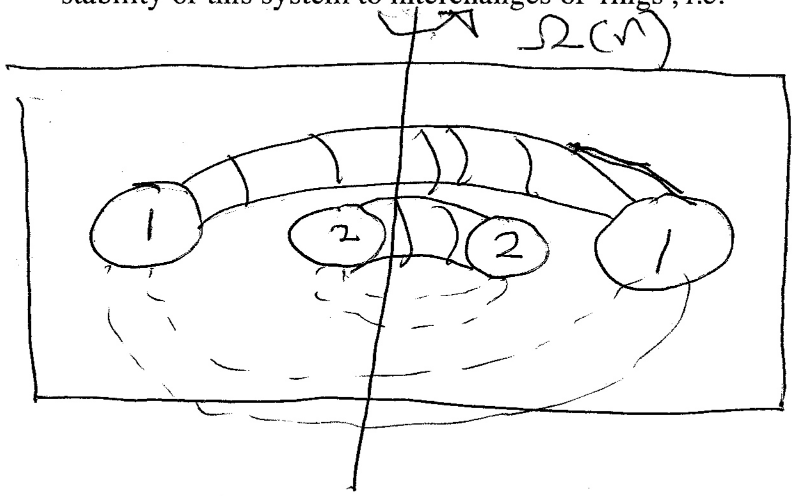
b.) Now, include thermal diffusivity (χ) and viscosity (ν) in your analysis. Calculate the critical temperature gradient for instability, assuming $\chi \sim \nu$. Discuss how this compares to the ideal limit. What happens if $\nu > \chi$?

2.) Now again, consider the system of Problem 2, now immersed in a uniform magnetic field $\underline{B} = B_0\hat{z}$.

a.) Assuming ideal dynamics, use the Energy Principle to analyze the stability of a convection cell of vertical wavelength k_z. Of course, $k_z L_p >> 1$, where L_p is a mean pressure scale length. What is the effect of the magnetic field? Can you estimate how the growth rate changes?

b.) Now, calculate the growth rate using the full MHD equations. You may assume $\underline{\nabla} \cdot \underline{V} = 0$. What structure convection cell is optimal for vertical transport of heat when B_0 is strong? Explain why. What happens when $B_0 \to \infty$? Congratulations - you have just derived a variant of the Taylor-Proudman theorem!

3.) Consider a rotating fluid with mean $\underline{V} = r\Omega(r)\hat{\theta}$. Your task here is to analyze the stability of this system to interchanges of 'rings', i.e.



In all cases, assume $\underline{\nabla} \cdot \underline{V} = 0$ and $k_\theta = 0$, so the interchange motions carry no angular momentum themselves and the cells sit in the r-z plane.

a.) At the level of a "back-of-an-envelope" calculation, calculate the change in energy resulting from the incompressible interchange of rings (1) and (2). Note that $E = L^2/2mr^2$ and that the angular momentum L of an interchanged ring is conserved, since $k_\theta = 0$. From this, what can you conclude about the profile of $\Omega(r)$ necessary for stability? Congratulations - you have just derived the Rayleigh criterion!

b.) Now, calculate the interchange growth rate by a direct solution of the fluid equations. You may find it helpful to note that for rotating fluids in cylindrical geometry:

$$\frac{\partial V_r}{\partial t} + \underline{V} \cdot \underline{\nabla} V_r - \frac{V_\theta^2}{r} = \frac{-1}{\rho}\frac{\partial P}{\partial r},$$

$$\frac{\partial V_\theta}{\partial t} + \underline{V} \cdot \underline{\nabla} V_\theta + \frac{V_r V_\theta}{r} = \frac{-1}{\rho r}\frac{\partial P}{\partial \theta},$$

$$\frac{\partial V_z}{\partial t} + \underline{V} \cdot \underline{\nabla} V_z = \frac{-1}{\rho}\frac{\partial P}{\partial z}.$$

Show that you recover the result of part (a).

c.) Compare and contrast this interchange instability to an incompressible Rayleigh-Taylor instability. Make a table showing the detailed correspondences.

4.) This problem asks you to explore the Current Convective Instability (CCI) in a homogeneous medium and its sheared field relative, the Rippling Instability.

a.) Consider first a current carrying plasma in a straight magnetic field $\underline{B} = B_0 \hat{z}$ - i.e. ignore the poloidal field, etc. Noting that the resistivity η is a function of temperature (ala' Spitzer - c.f. Kulsrud 8.7), calculate the electrostatic resistive instability growth rate, assuming T evolves according to:

$$\frac{\partial T}{\partial t} + \underline{v} \cdot \underline{\nabla} T - \chi_{\parallel} \partial_z^2 T - \chi_{\perp} \nabla_{\perp}^2 T = 0$$

and the electrostatic Ohm's Law is just

$$-B_0 \partial_z \phi = \frac{1}{\eta} \frac{d\eta}{dt} \tilde{T} (\eta J_0).$$

b.) *Thoroughly* discuss the physics of this simple instability, i.e.

- what is the free energy source?
- what is the mechanism?
- what are the dampings and how do they restrict the unstable spectrum?
- how does spectral asymmetry enter?
- what is the cell structure?

c.) Use quasilinear theory and the wave breaking limit to estimate the heat flux from the C.C.I.

d.) Now, consider the instability in a *sheared* magnetic field.

i.) What difficulties enter the analysis?

ii.) Resolve the difficulty by considering coupled evolution of vorticity, Ohm's Law (in electrostatic limit but with temperature fluctuations) and electron temperature. Compute the growth rate in the limit $\chi_{\parallel},\ \chi_{\perp} \to 0$. Compute the mode width. Discuss how asymmetry enters here. Explain why.

e.) Noting that $\chi_{\parallel} >> \chi_{\perp}$ (why? - see Kulsrud 8.7), estimate when parallel thermal conduction becomes an important damping effect. Can $\chi_{\parallel}$ alone ever absolutely stabilize the rippling mode?

f.) Calculate the quasilinear heat flux and use the breaking limit to estimate its magnitude.

5a.) Derive the quasilinear equation for the evolution of the pressure profile in a plasma which supports resistive interchange turbulence. Express your answer in terms of the radial displacement spectrum $\left|\tilde{\varepsilon}_{r\underline{k}}\right|^2$.

b.) Assuming localized heating with on-axis central deposition, derive an expression for the stationary pressure gradient in terms of the intensity profile of the resistive interchange displacement spectrum.

c.) Using the wave-breaking estimate for saturation level $\left(\tilde{\varepsilon}_r \sim k_r^{-1}\right)$, calculate how β scales with input power, assuming fixed density. When does this scaling fail? Why?

6.) Read K.V. Roberts and J.B. Taylor, *Physics of Fluids*, Vol. 8, pg 315, 1965. Write a short essay (including cartoons and schematic equations) explaining the "point" of this paper. Contrast the "twisted slicing modes" it describes with the resistive interchange mode discussed in class.